

# THE EULER AND NAVIER-STOKES EQUATIONS ON THE HYPERBOLIC PLANE

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**ABSTRACT.** We show that non-uniqueness of the Leray-Hopf solutions of the Navier-Stokes equation on the hyperbolic plane  $\mathbb{H}^2$  observed in [CC] is a consequence of the Hodge decomposition. We show that this phenomenon does not occur on  $\mathbb{H}^n$  whenever  $n \geq 3$ . We also describe the corresponding general Hamiltonian setting of hydrodynamics on complete Riemannian manifolds, which includes the hyperbolic setting.

## INTRODUCTION

Consider the initial value problem for the Navier-Stokes equations on a complete  $n$ -dimensional Riemannian manifold  $M$

$$\begin{aligned} (1) \quad & \partial_t v + \nabla_v v - Lv = -\text{grad } p, \quad \text{div } v = 0 \\ (2) \quad & v(0, x) = v_0(x). \end{aligned}$$

The symbol  $\nabla$  denotes the covariant derivative and  $L = \Delta - 2r$  where  $\Delta$  is the Laplacian on vector fields and  $r$  is the Ricci curvature of  $M$ . Dropping the linear term  $Lv$  from the first equation in (1) leads to the Euler equations of hydrodynamics

$$(3) \quad \partial_t v + \nabla_v v = -\text{grad } p, \quad \text{div } v = 0.$$

Most of the work on well-posedness of the Navier-Stokes equations has focused on the cases where  $M$  is either a domain in  $\mathbb{R}^n$  or the flat  $n$ -torus  $\mathbb{T}^n$ . In fundamental contributions J. Leray and E. Hopf established existence of an important class of weak solutions described as those divergence-free vector fields  $v$  in  $L^\infty([0, \infty), L^2) \cap L^2([0, \infty), H^1)$  which solve the Navier-Stokes equations in the sense of distributions and satisfy

$$(4) \quad \|v(t)\|_{L^2}^2 + 4 \int_0^t \|\text{Def } v(s)\|_{L^2}^2 ds \leq \|v_0\|_{L^2}^2 \quad \text{and} \quad \lim_{t \searrow 0} \|v(t) - v_0\|_{L^2} = 0$$

for any  $0 \leq t < \infty$  and where  $\text{Def } v = \frac{1}{2}(\nabla v + \nabla v^T)$  is the so-called deformation tensor. When  $n = 2$  using interpolation inequalities and energy estimates it is possible to show that the Leray-Hopf solutions are unique and regular but the problem is in general open for  $n = 3$ , see e.g. [CF] or [MB].

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There have also been studies on curved spaces, which with few exceptions have been confined to compact manifolds (possibly with boundary), see e.g. [Ta] and the references therein. In a recent paper Chan and Czubak [CC] studied the Navier-Stokes equation on the hyperbolic plane  $\mathbb{H}^2$  and more general non-compact manifolds of negative curvature. In particular, using the results of Anderson [An] and Sullivan [Su] on the Dirichlet problem at infinity, they showed that in the former case the Cauchy problem (1)-(2) admits non-unique Leray-Hopf solutions.

Our goal in this note is to provide a direct formulation of the non-uniqueness of the Leray-Hopf solutions on  $\mathbb{H}^2$  which turns out to rely on the specific form of the Hodge decomposition for 1-forms (or vector fields) in this case. We also show that no such phenomenon can occur in the hyperbolic space  $\mathbb{H}^n$  with  $n \geq 3$ . As a by-product, we describe the corresponding Hamiltonian setting of the Euler equations on complete Riemannian manifolds (in particular, hyperbolic spaces).

We point out that this type of non-uniqueness cannot be found in the Euler equations. Furthermore, it is of a different nature than the examples constructed e.g., by Shnirelman [Sh] or De Lellis and Székelyhidi [DS]. On the other hand, it is similar to non-uniqueness of solutions of the Navier-Stokes equations defined in unbounded domains of the higher-dimensional Euclidean space, cf. Heywood [He].

## 1. STATIONARY HARMONIC SOLUTIONS OF THE EULER EQUATIONS

Our main result is summarized in the following theorem.

### Theorem 1.1.

- (i) *There exists an infinite-dimensional space of stationary  $L^2$  harmonic solutions of the Euler equations on  $\mathbb{H}^2$ .*
- (ii) *There are no stationary  $L^2$  harmonic solutions of the Euler equations on  $\mathbb{H}^n$  for any  $n > 2$ .*

*Proof.* Recall the Hamiltonian formulation of the Euler equations (3) on a complete Riemannian manifold  $M$ , see e.g. [AK]. Consider the Lie algebra  $\mathfrak{g}_{\text{reg}} = \text{Vect}_\mu(M)$  of (sufficiently smooth) divergence-free vector fields on  $M$  with finite  $L^2$  norm. Its dual space  $\mathfrak{g}_{\text{reg}}^*$  has a natural description as the quotient space  $\Omega_{L^2}^1 / \overline{d\Omega_{L^2}^0}$  of the  $L^2$  1-forms modulo (the  $L^2$  closure of) the exact 1-forms on  $M$ . Namely, the pairing between cosets  $[\beta] \in \Omega_{L^2}^1 / \overline{d\Omega_{L^2}^0}$  of 1-forms  $\beta \in \Omega_{L^2}^1$  and vector fields  $w \in \text{Vect}_\mu(M)$  is given by

$$\langle [\beta], w \rangle := \int_M (\iota_w \beta) d\mu,$$

where  $\iota_w$  is the contraction of a differential form with a vector field  $w$ , and  $\mu$  is the Riemannian volume form on  $M$ .

Let  $A : \mathfrak{g}_{\text{reg}} \rightarrow \mathfrak{g}_{\text{reg}}^*$  denote the inertia operator defined by the Riemannian metric. The operator  $A$  assigns to a vector field  $v \in \text{Vect}_\mu(M)$  the coset  $[v^\flat]$  of the corresponding 1-form  $v^\flat$  via the pairing given by the metric. The coset is defined as the 1-form up

to adding differentials of the  $L^2$  functions on  $M$ . Thus, in the Hamiltonian framework the Euler equation reads

$$\frac{d}{dt}[v^\flat] = -L_v[v^\flat],$$

where  $[v^\flat] \in \Omega_{L^2}^1 / \overline{d\Omega_{L^2}^0}$  and  $L_v$  is the Lie derivative in the direction of the vector field  $v$ .

The space  $\Omega_{L^2}^1$  of the  $L^2$  1-forms on a complete manifold  $M$  admits the Hodge-Kodaira decomposition

$$\Omega_{L^2}^1 = \overline{d\Omega_{L^2}^0} \oplus \overline{\delta\Omega_{L^2}^2} \oplus \mathcal{H}_{L^2}^1,$$

where the first two summands denote the  $L^2$  closures of the images of the operators  $d$  and  $\delta$ , while  $\mathcal{H}_{L^2}^1$  is the space of the  $L^2$  harmonic 1-forms on  $M$ . Therefore, we have a natural representation of the dual space

$$\mathfrak{g}_{\text{reg}}^* = \overline{\delta\Omega_{L^2}^2} \oplus \mathcal{H}_{L^2}^1.$$

It turns out that the summand of the harmonic forms in the above representation corresponds to steady solutions of the Euler equation. Namely, one has the following proposition.

**Proposition 1.2.** *Each harmonic 1-form on a complete manifold  $M$  which belongs to  $L^2 \cap L^4$  defines a steady solution of the Euler equation (3) on  $M$ .*

*Proof of Proposition 1.2.* Let  $\alpha$  be a bounded  $L^2$  harmonic 1-form on  $M$ . Let  $v_\alpha$  denote the divergence-free vector field corresponding to  $\alpha$ , i.e.,  $v_\alpha^\flat = \alpha$ . Since the 1-form  $\alpha$  is harmonic, using Cartan's formula gives

$$\frac{d}{dt}\alpha = -L_{v_\alpha}\alpha = -\iota_{v_\alpha}d\alpha - d\iota_{v_\alpha}\alpha = -d\iota_{v_\alpha}\alpha.$$

We claim that  $\iota_{v_\alpha}\alpha \in \Omega_{L^2}^0$  and consequently  $d\iota_{v_\alpha}\alpha \in d\Omega_{L^2}^0$ . Indeed, by the definition of the vector field  $v_\alpha$  we have

$$\|\iota_{v_\alpha}\alpha\|_{L^2}^2 = \int_M (\alpha(v_\alpha))^2 d\mu = \|\alpha\|_{L^4}^4,$$

which is finite by assumption. It follows that the 1-form  $d\iota_{v_\alpha}\alpha$  must correspond to the zero coset in the quotient space  $\mathfrak{g}_{\text{reg}}^* = \Omega_{L^2}^1 / \overline{d\Omega_{L^2}^0}$ , which in turn implies that  $\frac{d}{dt}\alpha = 0 \in \mathfrak{g}_{\text{reg}}^*$ . The latter means that the 1-form  $\alpha$  defines a steady solution of the Euler equation, which proves the proposition.  $\square$

If  $M$  is compact then the space of harmonic 1-forms is always finite-dimensional (and isomorphic to the deRham cohomology group  $H^1(M)$ ). According to a well-known result of Dodziuk [Do], the hyperbolic space  $\mathbb{H}^n$  carries no  $L^2$  harmonic  $k$ -forms except for  $k = n/2$ , in which case it is infinite-dimensional. Therefore, there can be no  $L^2$  harmonic stationary solutions of the Euler equations on  $\mathbb{H}^n$  for any  $n > 2$ , which proves part (ii) of the theorem.

To prove part (i) we note that for  $n = 2$  the space of harmonic 1-forms on  $\mathbb{H}^2$  is infinite-dimensional. Moreover, it allows for the following construction. Consider the

subspace  $\mathcal{S} \subset \mathcal{H}_{L^2}^1$  of 1-forms which are differentials of bounded harmonic functions whose differentials are in  $L^2$

$$\mathcal{S} = \{d\Phi \mid \Phi \text{ is harmonic on } \mathbb{H}^2 \text{ and } d\Phi \in L^2\}.$$

It turns out that the subspace  $\mathcal{S}$  is already infinite-dimensional. Indeed, let us consider the Poincaré model of  $\mathbb{H}^2$ , i.e., the unit disk  $\mathbb{D}$  with the hyperbolic metric  $\langle \cdot, \cdot \rangle_h$ , which we denote by  $\mathbb{D}_h$ . It is conformally equivalent to the standard unit disk with the Euclidean metric  $\langle \cdot, \cdot \rangle_e$ , denoted by  $\mathbb{D}_e$ . Bounded harmonic functions on  $\mathbb{D}_h$  can be obtained by solving the Dirichlet problem on  $\mathbb{D}_e$ , i.e., by constructing harmonic functions  $\Phi$  on  $\mathbb{D}$  with boundary values  $\varphi$  prescribed on  $\partial\mathbb{D}$ . First, the 1-form  $d\Phi$  is clearly harmonic:

$$\Delta d\Phi = d\delta d\Phi = d\Delta\Phi = 0.$$

Secondly, observe that

$$\begin{aligned} \|d\Phi\|_{L^2(\mathbb{D}_h)}^2 &= \int_{\mathbb{D}} \langle d\Phi, d\Phi \rangle_h d\mu_h = \int_{\mathbb{D}} \det(g^{ij}) \langle d\Phi, d\Phi \rangle_e \det(g_{ij}) d\mu_e \\ &= \int_{\mathbb{D}} \langle d\Phi, d\Phi \rangle_e d\mu_e = \|d\Phi\|_{L^2(\mathbb{D}_e)}^2, \end{aligned}$$

and

$$\begin{aligned} \|d\Phi\|_{L^4(\mathbb{D}_h)}^4 &= \int_{\mathbb{D}} \langle d\Phi, d\Phi \rangle_h^2 d\mu_h = \int_{\mathbb{D}} \det^2(g^{ij}) \langle d\Phi, d\Phi \rangle_e^2 \det(g_{ij}) d\mu_e \\ &= \int_{\mathbb{D}} (1 - |z|^2)^2 \langle d\Phi, d\Phi \rangle_e^2 d\mu_e(z) \leq \int_{\mathbb{D}} \langle d\Phi, d\Phi \rangle_e^2 d\mu_e = \|d\Phi\|_{L^4(\mathbb{D}_e)}^4, \end{aligned}$$

where  $\det(g_{ij}) = 1/(1 - |z|^2)^2$  is the determinant of the hyperbolic metric.

Furthermore, for sufficiently smooth boundary values  $\varphi \in C^{1+\alpha}(\partial\mathbb{D})$  there is a uniform upper bound for its harmonic extension inside the disk:

$$|d\Phi(x)| \leq C \|\varphi\|_{C^{1+\alpha}(\partial\mathbb{D})}$$

for any  $x \in \mathbb{D}$  and  $0 < \alpha < 1$ , and some positive constant  $C$ , see e.g. [GT]. This implies that (for sufficiently smooth  $\varphi$ ) the 1-forms  $d\Phi$  define an infinite-dimensional subspace  $\mathcal{S}$  of harmonic forms in  $L^2 \cap L^4$ , which satisfy assumptions of the proposition above. It follows that they define an infinite-dimensional space of stationary solutions of the Euler equations on the hyperbolic plane  $\mathbb{H}^2$ . This completes the proof of Theorem 1.1.  $\square$

## 2. NON-UNIQUE LERAY-HOPF SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

Using the fact that suitably rescaled steady solutions of the Euler equations also solve the Navier-Stokes system the authors in [CC] obtained a type of ill-posedness result for the Leray-Hopf solutions in the hyperbolic setting.

**Theorem 2.1** ([CC]). *Given a vector field  $v_e = (d\Phi)^\sharp$  on  $\mathbb{H}^2$  there exist infinitely many real-valued functions  $f(t)$  for which  $v_{ns} = f(t)v_e$  is a weak solution of the Navier-Stokes equations with decreasing energy (i.e., satisfying the Leray-Hopf conditions).*

An immediate consequence of this result and Theorem 1.1 is the following

**Corollary 2.2.** *There exist infinitely many weak Leray-Hopf solutions to the Navier-Stokes equation on  $\mathbb{H}^2$ . There are no non-unique Leray-Hopf harmonic solutions to the Navier-Stokes equation on  $\mathbb{H}^n$  with  $n \geq 3$  arising from the above construction.*

**Remark 2.3.** The phenomenon of nonuniqueness of solutions to the Navier-Stokes equation in unbounded domains  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , of higher-dimensional Euclidean spaces is of similar nature, see [He]. Indeed, that construction is based on the existence of a harmonic function with gradient in  $L^2$  and appropriate boundary conditions in such domains. The Green function  $\Phi(x) = G(a, x)$  centered at a point  $a$  outside of  $\Omega$  has the decay like  $G(a, x) \sim |x|^{2-n}$  as  $x \rightarrow \infty$ , so that  $|d\Phi(x)| \sim |x|^{1-n}$  and hence  $|d\Phi(x)|^2 \sim |x|^{2-2n}$ . Thus, for  $n \geq 3$  the 1-forms  $d\Phi$  belong to  $L^2 \cap L^4$  on  $\Omega$ . The corresponding divergence-free vector fields  $(d\Phi)^\sharp$  provide examples of stationary Eulerian solutions in  $\Omega$  (with nontrivial boundary conditions) and can be used to construct time-dependent weak solutions  $v_{ns} = f(t)(d\Phi)^\sharp$  to the Navier-Stokes equation in  $\Omega$ , as in Theorem 2.1.

### 3. APPENDIX

To make this note self-contained we provide here some details of the construction of the weak solutions given in [CC]. It will be convenient to rewrite the Navier-Stokes equations (1) in the language of differential forms

$$(5) \quad \partial_t v^\flat + \nabla_v v^\flat - \Delta v^\flat + 2r(v^\flat) = -dp, \quad \delta v^\flat = 0$$

where  $\delta v^\flat = -\operatorname{div} v$  and  $\Delta v^\flat = d\delta v^\flat + \delta d v^\flat$  is the Laplace-deRham operator on 1-forms.

Let  $v$  be the vector field  $v_{ns} = f(t)(d\Phi)^\sharp$  on  $\mathbb{H}^2$  as in Theorem 2.1. Since the 1-form  $d\Phi$  is harmonic one only needs to compute the covariant derivative term and the Ricci term:

$$\nabla_{v_{ns}} v_{ns}^\flat = \frac{1}{2} f^2(t) d|d\Phi|^2 \quad \text{and} \quad 2r(v_{ns}^\flat) = -2f(t)d\Phi.$$

Direct computation, taking into account the fact that for  $\mathbb{H}^2$  we have  $r = -1$ , shows that both terms can be absorbed by the pressure term, so that the pair  $(v_{ns}^\flat, p)$ , where  $p := (2f(t) - f'(t))\Phi - 1/2 f^2(t)|d\Phi|^2$  satisfies the equations (5).

Finally, a quick inspection shows that any differentiable function  $f(t)$  satisfying

$$f^2(t) + 4 \int_0^t f^2(s) ds \leq f^2(0)$$

yields a vector field  $v_{ns}$  which satisfies the remaining conditions in (4) required of a Leray-Hopf solution.

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